Thompson’s group $\mathcal{T}$ as automorphism group of a cellular complex

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**Tits (1970)**

Most subgroups of automorphism groups of trees generated by vertex stabilizers are simple.

**Haglund and Paulin (1998)**

Most automorphism groups of negatively curved polyhedral complexes are virtually simple.

**We are going to be interested in the reciprocal situation:**

Given a simple group $G$ (in our case Thompson’s group $\mathcal{T}$), find an interesting cellular complex $C$ such that its automorphism group is ‘essentially’ the group $G$. 

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$\mathcal{T}$ as automorphism group
**Introduction**

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### Goal of the talk

What do we already know about $\mathcal{T}$ acting on complexes?

#### Farley, 2005

$\mathcal{F}$, $\mathcal{T}$ and $\mathcal{V}$ act properly and isometrically on CAT(0) cubical complexes.

#### F.-Nguyen, 2011

There exists a cellular complex $\mathcal{C}$ such that $\Aut_+(\mathcal{C}) \simeq \mathcal{T}$, where $\Aut_+ = \text{subgroup of orientation-preserving automorphisms}$.  

Where does the complex $\mathcal{C}$ come from?


There exists a planar surface $\Sigma$ of infinite type which has Thompson’s group $\mathcal{T}$ as asymptotic mapping class group.
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There exists a planar surface $\Sigma$ of infinite type which has Thompson’s group $\mathcal{T}$ as asymptotic mapping class group.
Another viewpoint

- $\Sigma_{g,n}$: compact, connected, orientable surface of genus $g$ with $n$ marked points.
- $\text{MCG}^*(\Sigma_{g,n})$: extended mapping class group of $\Sigma_{g,n}$.
- $C_c$: curve complex of $\Sigma_{g,n}$.
- $C_p$: pants complex of $\Sigma_{g,n}$.

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**Ivanov-Korkmaz, 1997**

\[ \text{MCG}^*(\Sigma_{g,n}) \cong \text{Aut}(C_c), \text{ unless } g = 0 \text{ and } n \leq 4, \text{ or } g = 1 \text{ and } n \leq 2, \text{ or } g = 2 \text{ and } n = 0. \]

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The surface $\Sigma$

The triangulation $E$ of $\mathbb{D}^2$ and its dual tree
The surface $\Sigma$ and its hexagonal tessellation
Introduction

Surface $\Sigma$

AMCG

Complex $\mathbb{C}$

Aut($\mathbb{C}$)

The surface $\Sigma$

Dyadic rational numbers at the closure of $\partial \Sigma$
The asymptotic mapping class group of $\Sigma$

**Separating arcs: example**
The asymptotic mapping class group of $\Sigma$

Asymptotically rigid homeomorphism: example
The asymptotic mapping class group of $\Sigma$

Homeo_{a}(\Sigma): asymptotically rigid homeomorphisms of $\Sigma$.

**Definition**

AMCG(\Sigma): quotient of Homeo_{a}(\Sigma) by the group of isotopies of $\Sigma$.

**Proposition**

AMCG(\Sigma) is isomorphic to Thompson’s group $\mathcal{T}$.

**Definition**

Thompson’s group $\mathcal{T}$ is the set of orientation preserving piecewise linear homeomorphisms of the circle $S^1 = [0, 1]/\sim$ such that:

1. the points of non-differentiability are dyadic rational numbers,
2. the derivatives (where defined) are powers of 2, and
3. the set of dyadic rational numbers is fixed.
The asymptotic mapping class group of $\Sigma$

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**Idea of the proof**
The asymptotic pants complex $C$: definition

**Vertices of $C$**
The asymptotic pants complex $C$: definition

Edges of $C$
The asymptotic pants complex $\mathcal{C}$: definition

Edges of $\mathcal{C}$
The asymptotic pants complex $C$: definition

Squared 2-cells of $C$
Pentagonal 2-cells of $C$
The asymptotic pants complex $C$: definition

Some properties of $C$

1. $C$ is connected and simply connected.

2. $C$ is locally infinite. 
   \{neighbours of the vertex $v}\leftrightarrow\{arcs of the triangulation $v\}.

3. $C$ is not Gromov hyperbolic.
The asymptotic pants complex $C$: definition

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The action of $\mathcal{T}$ on $C$

$\mathcal{T}$ acts on $C$

**Proposition**

$\mathcal{T}$ acts transitively on $C$ by automorphisms. Furthermore, the map $\Psi : \mathcal{T} \rightarrow \text{Aut}(C)$ is injective.

**Action:**

![Diagram showing the action of $\mathcal{T}$ on $C$]
The action of $\mathcal{T}$ on $C$

Transitivity of the action

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$\mathcal{T}$ as automorphism group
The action of $T$ on $C$

**Construction of** $f \in T$ **such that** $f \cdot E = v$. 

![Diagram with trivalent graphs and curves](image)
The action of $\mathcal{T}$ on $C$

Construction of $f \in \mathcal{T}$ such that $f \cdot E = v$. 

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**Construction of $f \in \mathcal{T}$ such that $f \cdot E = v$.**
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$f \in \mathcal{T}$ with $f \cdot E = \nu$. 

\[ f \in \mathcal{T} \text{ with } f \cdot E = \nu. \]
Link complex

**Definition**

Let $v \in C^0$. The **link complex** $\mathcal{L}^2(v)$ of $v$ is the simplicial complex

- **Vertices**: Neighbours of $v$ in $C$.
- **Edges**: $u$ and $w$ are joined if $u, v, w$ lie in a pentagonal 2-cell.
- **2-simplex**: Triangles in the 1-skeleton $\mathcal{L}^1(v)$.
Link complex

Automorphisms of $\mathcal{C}$ induce link isomorphisms

Lemma

Let $v \in \mathcal{C}^0$. Then, every automorphism $\varphi \in \text{Aut}(\mathcal{C})$ induces an isomorphism $\varphi_{*,v} : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$, where $w = \phi(v)$.

Question 1: what about the reciprocal?

Given $v$ and $w$ be vertices of $\mathcal{C}$ and $i : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$ isomorphism, can we always find $\varphi \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$?

NO.

Question 2: why not?

Which is the main obstruction to the existence of this automorphism?
Automorphisms of $\mathcal{C}$ induce link isomorphisms

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Question 2: why not?

Which is the main obstruction to the existence of this automorphism?
Main obstruction to extension of link isomorphisms

Proposition

Let $i : \mathcal{L}^2(v) \rightarrow \mathcal{L}^2(w)$ be a link isomorphism such that:
- it is orientation reversing on $\Delta_1 = (u_0, u_1, u_2)$, and
- it is orientation preserving on $\Delta_2 = (u_0, u_3, u_4)$.

Then, there does not exist $\varphi \in \text{Aut}(C)$ with $\varphi_{*,v} = i$. 
Example of non-extensible link isomorphism
Lemma

Let \( v, w \in C^0 \), and \( i : L^2(v) \to L^2(w) \) \( v \)-orientation preserving isomorphism. Then, there exists a unique automorphism \( \varphi_i \in \text{Aut}(C) \) such that \( \varphi_{*,v} = i \).

Furthermore, \( \varphi_i \) is an element of \( \mathcal{T} \).

Remark

For all \( \varphi \in \mathcal{T} \leq \text{Aut}(C) \) and for all \( v \) vertex of \( C \), \( \varphi_{*,v} \) is \( v \)-orientation preserving.

Lemma

Let \( i_R : L^2 E \to L^2 E \) be the orientation reversing isomorphism obtained by the symmetry of axis \( I^1_0 \). Then, there exists a unique automorphism \( \varphi_R \in \text{Aut}(C) \) such that \( \varphi_{R*,E} = i_R \).
### Extension of link isomorphisms

**Lemma**

Let $v, w \in \mathcal{C}^0$, and $i : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$ $v$-orientation preserving isomorphism. Then, there exists a unique automorphism $\varphi_i \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$.

Furthermore, $\varphi_i$ is an element of $\mathcal{T}$.

**Remark**

For all $\varphi \in \mathcal{T} \leq \text{Aut}(\mathcal{C})$ and for all $v$ vertex of $\mathcal{C}$, $\varphi_{*,v}$ is $v$-orientation preserving.

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$\mathcal{T}$ as automorphism group
Introduction

Surface $\Sigma$

AMCG

Complex $C$

$\text{Aut}(C)$

Link isomorphisms and extensions

Structure of $\text{Aut}(C)$

**Theorem (F.-Nguyen, 2011)**

$\text{Aut}(C) \cong T \rtimes \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Consider

$$\Psi : \varphi \mapsto \begin{cases} 
0, & \text{if } \varphi \text{ is orientation preserving} \\
1, & \text{if } \varphi \text{ is orientation reversing.} 
\end{cases}$$

$\ker(\Psi) = T$.

Section: $1 \in \mathbb{Z}/2\mathbb{Z} \rightarrow \varphi_R$.

Thus, we have

$$1 \rightarrow T \rightarrow \text{Aut}(C) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1,$$

and it splits.
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Link isomorphisms and extensions

Structure of Aut($C$)

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Thanks for your attention.